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A METHOD OF SOLVING PROBLEMS OF THE LINEAR THEORY OF ELASTICITY*

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A method is developed for solving linear boundary value problems, based on their interpretation in the spirit of functional analysis. In the special case of the theory of elasticity, the stress and strain fields are considered as elements of a real Hilbert space of symmetric tensors of the second rank. On the basis of the second derivative of Green's tensor of the equilibrium equations, projection operators $\boldsymbol{P}$ and $\overline{Q}$ are constructed that satisfy the equation $\boldsymbol{P}+\overline{Q}=I$. The solution of the mixed boundary value problem is represented in the form of Neumann series, whose sufficient conditions for convergence are written in the form of operator inequalities which lend themselves to a simple interpretation in the language of energy functionals. By strengthening these conditions we can express them in terms of the closeness of the coefficients of the problem λ and λ_c. A representation of the potential energy is given in the form of a certain functional which can always be expanded in series. The limits within which the exact value of the potential energy lies is obtained.

The purpose of this paper is to develop a method for solving linear boundary value problem based on the formalism of Green's tensors on the one hand, and on the interpretation of these problems in the spirit of functional analysis, on the other. Consideration of the stress field σ and strain field ε as elements of a real Hilbert space H of symmetric tensors of the second rank and the introduction of the operators P and Q, constructed on the basis of Green's tensor G and acting in the space H, enable a transfer to be made from the equilibrium equation to a functional equation of the form (1.13). The iteration method often utilized to solve such equations results in a solution in the form of the Neumann series (1.15) whose convergence conditions are not always evident.

It is assumed that the elastic properties of the medium under investigation are described by a symmetric fourth-rank tensor $\lambda=\lambda(\mathbf{r})$. (Here and henceforth, the tensor subscripts are omitted, for simplicity, almost everywhere, and the vector quantities are denoted by heavy type. In the product A_kB_l of the tensor A_k of rank k and the tensor B_l of rank l the summation is over all subscripts of the tensor B_l if $l<k$ and over the n inner subscripts of the tensors A_k and B_l if $k=l=2n$).

A medium for which the solution of the initial problem is known is used as the auxiliary medium (the comparison medium). Its elastic properties are described by the tensor λ_c. Without limiting the generality, we consider λ and λ_c symmetric operators (see Sect.2). This enables the method to be extended to viscoelastic media and a medium with a microstructure. Important relations are obtained in Sect.2 for the operators P and Q and their associated $\boldsymbol{P},\overline{Q}$. It is shown that these belong to the class of projections. This circumstance exerts a substantial influence on the form of the convergence conditions for series (2.14). By rounding off the sufficient conditions for (3.4) and (3.5) to converge, we obtain conditions (3.7) and (3.8) (or (3.11)), which when $\lambda_c\mu_c\neq I$ turn out to be independent and can be satisfied simultaneously.

A representation of the potential energy U in the form of functionals computed using the auxiliary fields σ_c and ε_c referred to the comparison medium, is given in Sect.4. It is shown in Sect.5 that the energy u' is representable in the form of the series (5.1) or (5.2) whose sign-definiteness depends on the properties of the functionals l_k or m_k, respectively. In any case the limits within which the exact value of u' lies can be computed.

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1. Consider the equilibrium equation for an arbitrary linear elastic medium with the boundary conditions

$$Lu = -f,\ L = \operatorname{div} \lambda \operatorname{def} \equiv \nabla\lambda\nabla,\ \mathbf{r} \in V \tag{1.1}$$

$$\mathbf{u} = \mathbf{u}_0,\ \mathbf{r} \in S_1;\ \mathbf{t} = \mathbf{t}_0,\ \mathbf{r} \in S_2 \tag{1.2}$$

$$\mathbf{t} = \sigma\mathbf{n},\ \sigma = \lambda\varepsilon,\ S_1 \cup S_2 = S$$

Here $\mathbf{u}$ is the displacement vector, $\mathbf{f}$ is the vector of the volume density of the external forces, and $\mathbf{n}$ is the unit vector of the external normal to the surface S bounding the volume V of this medium. The strain tensor ε is related to the displacement vector by the relationship $\varepsilon = \operatorname{def} \mathbf{u}$, as a result of which it satisfies the compatibility equation /1/

$$\operatorname{Ink} \varepsilon = 0,\ \operatorname{Ink}_{ijkl} = e_{ipk} e_{jql} \nabla_p \nabla_q \tag{1.3}$$

where summation is over the subscripts that appear twice.

In addition to (1.1)-(1.3), we will assume that there are analogous equations for the comparison medium to which the transition is made by replacing λ by λ_c. The fields corresponding to this medium are denoted by the additional subscript c.

We will now find the relation between the fields ε and ε_c. To this end we introduce the difference field $\mathbf{u}_1 = \mathbf{u} - \mathbf{u}_c \equiv \mathbf{u}'$, to which the stresses $\sigma_1 = \lambda_c \varepsilon_1$ correspond in the medium with the elastic properties λ_c, where $\varepsilon_1 = \operatorname{def} \mathbf{u}_1 = \varepsilon'$. Evidently $\sigma_1 \neq \sigma' = \sigma - \sigma_c$. Using the polarization stress tensor τ /1/, we can write

$$\sigma' = \sigma_1 + \tau = \lambda_c \varepsilon' + \tau \tag{1.4}$$

Subtracting the equation and boundary conditions for the comparison medium, respectively, from (1.1) and the boundary conditions (1.2), and taking into account that the external effects are identical in both cases, we obtain an equation with the boundary conditions

$$L_c \mathbf{u}_1 = -\mathbf{f}_1,\quad \mathbf{f}_1 = \nabla\tau,\quad \mathbf{r} \in V \tag{1.5}$$

$$\mathbf{u}_1 = 0,\quad \mathbf{r} \in S_1;\quad \mathbf{t}_1 = -\tau\mathbf{n},\quad \mathbf{r} \in S_2$$

We will find the solution of problem (1.5) by using Green's tensor $G(\mathbf{r}, \mathbf{r}_1)$ that satisfies the equation /2, 3/ and homogeneous boundary conditions

$$L_c G(\mathbf{r}, \mathbf{r}_1) = -\delta(\mathbf{r} - \mathbf{r}_1);\quad \mathbf{r}, \mathbf{r}_1 \in V \tag{1.6}$$

$$G(\mathbf{r}, \mathbf{r}_1) = 0,\quad \mathbf{r} \in S_1;\quad T(\mathbf{r}, \mathbf{r}_1) = 0,\quad \mathbf{r} \in S_2;\quad \mathbf{r}_1 \in V$$

$$T_{ij}(\mathbf{r}, \mathbf{r}_1) = n_p \lambda_{ipkl}^c \nabla_k G_{lj}(\mathbf{r}, \mathbf{r}_1)$$

We will change to the polarization stresses τ in the integrands in the general solution of problem (1.5), and we will use Gauss's theorem. We obtain

$$u_i'(\mathbf{r}) = -\int \tau_{jk}(\mathbf{r}_1) \nabla_k^1 G_{ij}(\mathbf{r}, \mathbf{r}_1)\, dV_1 + \Delta u_i'(\mathbf{r}) \tag{1.7}$$

$$\Delta u_i'(\mathbf{r}) = \int [G_{ji}(\mathbf{r}_1, \mathbf{r}) t_j'(\mathbf{r}_1) - T_{ji}(\mathbf{r}_1, \mathbf{r}) u_j'(\mathbf{r}_1)]\, dS(\mathbf{r}_1),\ \mathbf{t}' = \sigma'\mathbf{n}$$

where ∇^1 is the nabla operator for the coordinate $\mathbf{r}_1$.

The solution (1.7) contains the surface integral $\Delta u_i'(\mathbf{r})$ that equals zero within the domain and is not defined on the boundary S since its integral is zero at all points of the surface of integration, except the one where it becomes infinite. The limit values of this integral equal zero. Taking this remark into account, we will later write solution (1.7) without the term $\Delta u_i'(\mathbf{r})$.

The strain field

$$\varepsilon_{ij}' = \nabla_{(i} u'_{j)} = -\int \nabla_{(i} G_{j)(k, l_1)}(\mathbf{r}, \mathbf{r}_1) \tau_{kl}(\mathbf{r}_1)\, dV_1 \tag{1.8}$$

corresponds to the displacement field (1.7), where symmetrization is performed over the subscripts in parentheses, and the subscript l_1 denotes differentiation corresponding to the operator ∇_l^1.

Let us rewrite (1.8) in the form

$$\varepsilon' = Q\tau,\quad Q_{ijkl}(\mathbf{r}, \mathbf{r}_1) = -\nabla_{(i} G_{j)(k, l_1)}(\mathbf{r}, \mathbf{r}_1) \tag{1.9}$$

Here the integral operator Q and its kernel $Q(\mathbf{r}, \mathbf{r}_1)$ are denoted by the same letter. Substituting (1.9) into (1.4), we find

$$\sigma' = P\eta,\ \eta = -\mu_c\tau \tag{1.10}$$

where the integral operator P is related to Q by the relationships

$$-P = \lambda_c + \lambda_c Q \lambda_c,\quad -Q = \mu_c + \mu_c P \mu_c;\quad \lambda_c \mu_c = I \tag{1.11}$$

Formulas (1.9) and (1.10) agree in form with the analogous relationships in /4/, but they are obtained here in the more general case.

Expressing τ in (1.4) in terms of η, we obtain

$$\varepsilon' = \mu_c \sigma' + \eta \tag{1.12}$$

Here η is a polarization strain tensor playing a part analogous to τ in (1.4). The equations

$$\tau = \lambda'\varepsilon, \eta = \mu'\sigma; x' \equiv x - x_c, \lambda\mu = I$$

result from (1.4) and (1.12) and are used to reduce (1.9) and (1.10) to the form

$$\varepsilon = \varepsilon_c + Q\tau = \varepsilon_c + Q\lambda'\varepsilon, \quad \sigma = \sigma_c + P\eta = \sigma_c + P\mu'\sigma \tag{1.13}$$

The relationships obtained are inhomogeneous linear integral equations of the second kind /3/. In the case under consideration the domain of integration V is fixed, in which connection (1.13) are sometimes called integral equations of Fredholm type /3/, The solution of each enables the unknown fields σ and ε to be expressed in terms of the known σ_c and ε_c, which is equivalent to the solution of problem (1.1) and (1.2). Therefore, any of the equations (1.13) is equivalent to the problem (1.1), (1.2). (This fact lends itself to the consideration under which equations (1.13) are mutually independent. In this case, however, it is necessary to use two comparison media whose elastic properties are described by the tensors $\lambda_c^{(1)} \equiv \lambda_c$ and $\lambda_c^{(2)} \equiv \mu_c^{-1}$, respectively. Obviously, $\lambda_c\mu_c \neq I$). The displacement vector $\mathbf{u}$ can be found from (1.7).

On the other hand, relationships (1.13) can be interpreted as functional equations in a real Hilbert space H of symmetric second-rank tensors. The solution of equations (1.13) is here reduced to the problem of seeking operators a and b in the form

$$a = (I - X)^{-1}, \ X = Q\lambda'; \ b = (I - Y)^{-1}, \ Y = P\mu' \tag{1.14}$$

which are representable, under certain conditions, in the form of Neumann series /3, 5/

$$a = \sum_{k=0}^{\infty} X^k, \quad b = \sum_{k=0}^{\infty} Y^k \tag{1.15}$$

Before investigating the conditions under which the expansions (1.15) are possible, we will obtain certain equalities for the operators P and Q.

2. We define the scalar product of two elements ε and σ of the space H denoted by (ε, σ) by the equality

$$(\varepsilon, \sigma) = \int \varepsilon_{ij}\sigma_{ij}\, dv, \quad dv \equiv \frac{dV}{V} \tag{2.1}$$

In this paper, symmetric (Hermitian) operators are utilized, i.e., those operators satisfying the equation $A^+ = A$, where the superscript plus denotes the conjugate operations which reduces to transposition, defined by the relationships /3, 5/

$$(\varepsilon_1, A\varepsilon_1) = (A^+\varepsilon_1, \varepsilon_2) \tag{2.2}$$

in the case of real fields under consideration.

Taking account of (2.2) and (1.1), the operator Q introduced in (1.9) can be represented in the form

$$Q = -\nabla M \nabla^+ \tag{2.3}$$

where M is an integral operator whose kernel is Green's tensor G.

We will show that the operator Q is symmetric. To this end, we operate with the operator "plus" on both sides of (2.3). This yields

$$Q^+ = -(\nabla M \nabla^+)^+ = -(\nabla^+)^+ M^+ \nabla^+$$

Taking into account the reciprocity theorem that has the meaning of a symmetry condition for the operator M, and also the equation $(\nabla^+)^+ = \nabla$, we obtain $Q^+ = Q$. Because of (1.11) the operator P is also symmertic. The symmetry condition is considered satisfied for λ and λ_c.

We will examine the scalar product (ε', σ'), which, by virtue of (1.9), (1.10) and (2.2), equals

$$(\varepsilon', \sigma') = (Q\tau, P\eta) = (\tau, Q^+P\eta) = (\tau, QP\eta)$$

On the other hand, the following equation holds

$$(\varepsilon', \sigma') = \frac{1}{V}\int \mathbf{u}'\mathbf{t}'\, ds - \int \mathbf{u}' \operatorname{div} \sigma'\, dv$$

in which the surface integral (exactly equal to zero because of the boundary conditions (1.5)) should generally be omitted in connection with the procedure for computing volume integrals containing the difference fields introduced in obtaining (1.8) from (1.7). The volume integral on the right side of this equation vanishes because of (1.5) written in the form $\nabla\sigma' = 0$ taking (1.4) into account. Therefore, we finally obtain

$$(\tau, QP\eta) = (\eta, PQ\tau) = 0 \tag{2.4}$$

Equations (2.4) enable important relationships for the operators P and Q to be derived. Substituting the operator P from (1.11) into (2.4) and expressing η in terms of τ using (1.10), we find

$$(\tau,\ Q\tau) = -(\tau,\ Q\lambda_c Q\tau) = -(\dot{Q}\tau,\ \lambda_c Q\tau) \tag{2.5}$$

or in operator form

$$Q + Q\lambda_c Q = 0 \tag{2.6}$$

Because the integral form $(\varepsilon',\lambda_c\varepsilon')$ is positive, (2.5) means that the operator Q is negative definite in the sense of the inequality $(\tau,\ Q\tau) < 0$). Substituting the operator Q from (1.11) into (2.4), we can write in an analogous manner

$$(\eta,\ P\eta) = -(\eta,\ P\mu_c P\eta) = -(P\eta,\ \mu_c P\eta) \tag{2.7}$$

or in operator form

$$P + P\mu_c P = 0 \tag{2.8}$$

Because of $(\sigma',\ \mu_c\sigma')$ is positive (2.7) means that the operator P is negative definite, i.e. $(\eta,\ P\eta) < 0$.

Two fields ε and $\sigma = \lambda\varepsilon$ belonging to the space H but being distinguishable by the dimensional factor λ are in the computations presented above. However, it is more convenient to be rid of these differences by making both fields identical in the dimensional sense. This can be achieved by multiplying the fields σ and ε by the symmetric positive operators $\mu_c^{1/2}$ and $\lambda_c^{1/2}$, respectively. Because λ_c is positive and symmetrical the representation $\lambda_c = (\lambda_c^{1/2})^2$ is single-valued /5/.

We will introduce the following notation for the fields and operators:

$$\bar{\sigma} = \mu_c^{1/2}\sigma,\quad \bar{\varepsilon} = \lambda_c^{1/2}\varepsilon;\quad \bar{P} = -\mu_c^{1/2}P\mu_c^{1/2},\quad \bar{Q} = -\lambda_c^{1/2}Q\lambda_c^{1/2}$$

Equations (1.13) take the form

$$\begin{aligned} &\bar{\varepsilon} = \bar{\varepsilon}_c - \bar{Q}\bar{\lambda}'\bar{\varepsilon},\quad \bar{\lambda}' = \mu_c^{1/2}\lambda'\mu_c^{1/2};\quad \lambda_c^{1/2}\mu_c^{1/2} = I \\ &\bar{\sigma} = \bar{\sigma}_c - \bar{P}\bar{\mu}'\bar{\sigma},\quad \bar{\mu}' = \lambda_c^{1/2}\mu'\lambda_c^{1/2};\quad \bar{\sigma}_c = \bar{\varepsilon}_c = \lambda_c^{1/2}\varepsilon_c \end{aligned} \tag{2.9}$$

From (1.11) we obtain

$$\bar{P} + \bar{Q} = I \tag{2.10}$$

and (2.6) and (2.8) become

$$\bar{P}^2 = \bar{P},\ \bar{Q}^2 = \bar{Q} \tag{2.11}$$

Because of (2.10) and (2.11) the positive symmetric operators $\bar{P}$ and $\bar{Q}$ possess the properties of projection operators /5/. Therefore, the space H is representable in the form of the sum $H_1 + H_2$ of two subspaces which are mutual orthogonal complements. The inequalities /5/

$$0 \leqslant \bar{P} \leqslant I,\ 0 \leqslant \bar{Q} \leqslant I \tag{2.12}$$

are valid for the projection operators $\bar{P}$ and $\bar{Q}$, where the left values hold when H_1 or H_2 consist of one zero element while the right values hold if H_1 or H_2 agree with H.

Finally, we note that the solutions of (2.9) have the form

$$\bar{\varepsilon} = \bar{a}\bar{\varepsilon}_c,\quad \bar{a} = \lambda_c^{1/2}a\mu_c^{1/2};\quad \bar{\sigma} = \bar{b}\bar{\sigma}_c,\quad \bar{b} = \mu_c^{1/2}b\lambda_c^{1/2} \tag{2.13}$$

where we will have, in the same way as for (1.14) and (1.15), for the operators $\bar{a}$ and $\bar{b}$

$$\begin{aligned} &\bar{a} = (I - \bar{X})^{-1} = \sum_{k=0}^{\infty}\bar{X}^k,\quad \bar{X} = -\bar{Q}\bar{\lambda}' \\ &\bar{b} = (I - \bar{Y})^{-1} = \sum_{k=0}^{\infty}\bar{Y}^k,\quad \bar{Y} = -\bar{P}\bar{\mu}' \end{aligned} \tag{2.14}$$

It is possible to represent $\bar{a}$ and $\bar{b}$ in the form of series, provided they converge, which we now investigate.

3. The uniform convergence (in the norm) of the series (2.14) is examined below. We shall say /5/ that the sequence $\bar{a}_{(n)}$ converges to $\bar{a}$ in the norm if $\|\bar{a}_{(n)} - \bar{a}\| \to 0$ as $u \to \infty$. As is well-known, the concept of a norm is directly related to the scalar product (2.1). By definition, we have /3, 5/ (A is a certain operator)

$$\|f\| = (f,f)^{1/2},\quad \|A\| = \sup_{f\neq 0}\frac{\|Af\|}{\|f\|},\quad f \in H \tag{3.1}$$

According to a Banach theorem /6/, the operator $(1 - \bar{X})$ has a continuous inverse operator $\bar{a}$ of the form (2.14) if the norm of the operator $\bar{X}$ satisfies the inequality $\|\bar{X}\| \leqslant k_1 < 1$. The necessary and sufficient condition for the series $\bar{a}$ from (2.14) to converge is /6/ compliance with the inequalities $\|\bar{X}^n\| \leqslant k_1 < 1$ for a certain n.

According to the definition (3.1), the condition of the Banach theorem (the convergence of the first series in (2.14)) can be written in the form

$$\| \bar{X}f \|^2 = (f, \bar{X}^+\bar{X}f) \leqslant k_1^2 (f, f) = k_1^2 \| f \|^2, \ 0 \neq f \in H \tag{3.2}$$

where, unlike $\bar{X}$, the operator $\bar{X}^+\bar{X}$ possesses the property of symmetry. Rewriting (3.2) in operator form and taking account of (2.11), we obtain

$$\bar{\lambda}'\bar{Q}\bar{\lambda}' \leqslant k_1^2 I < I, \tag{3.3}$$

Returning to the initial quantities, we hence find

$$-\lambda' Q \lambda' \leqslant k_1^2 \lambda_c, \ 0 \leqslant k_1 < 1 \tag{3.4}$$

Inequality (3.4) is the sufficient condition for the first series in (2.14) to converge, and therefore, the series a from (1.15) also. By using analogous reasoning, we write the sufficient condition for the second series in (2.14) and b from (1.15) to converge in the form of the operator equation

$$-\mu' P \mu' \leqslant k_2^2 \mu_c, \ 0 \leqslant k_2 < 1 \tag{3.5}$$

Relying on the property of the operator $\bar{Q}$ of the form $\bar{Q} \leqslant I$, we will have from (3.3)

$$\bar{\lambda}'\bar{Q}\bar{\lambda}' \leqslant \bar{\lambda}'^2 \leqslant k_1^2 I \tag{3.6}$$

It is seen that condition (3.6) results in the sufficient condition for the first series in (2.14) and (1.15) to converge in the form

$$(1 - k_1) \lambda_c \leqslant \lambda \leqslant (1 + k_1) \lambda_c, \ 0 \leqslant k_1 < 1 \tag{3.7}$$

In a similar manner the sufficient condition for the second series in (2.14) and (1.15) to converge can be written in the form

$$(1 - k_2) \mu_c \leqslant \mu \leqslant (1 + k_2) \mu_c, \ 0 \leqslant k_2 < 1 \tag{3.8}$$

It can be shown that the inequalities (3.7) are equivalent to the corresponding inequalities for the potential energies.

For simplicity, we will examine the case of homogeneous boundary conditions and no external forces $\mathbf{f}$. Then the potential energies $U(\varepsilon) = {}^1/_2 (\varepsilon, \lambda\varepsilon)$ and $U_c(\varepsilon_c) = {}^1/_2 (\varepsilon_c, \lambda_c\varepsilon_c)$ of the investigated and auxiliary media satisfy the following inequalities because of the theorem on the minimum of the potential energy

$$U(\varepsilon) \leqslant U(\varepsilon_c), \ U_c(\varepsilon_c) \leqslant U_c(\varepsilon) \tag{3.9}$$

Combining inequalities (3.9) and (3.7), we can write

$$(1 - k_1) U_c(\varepsilon_c) \leqslant U(\varepsilon) \leqslant (1 + k_1) U_c(\varepsilon_c) \tag{3.10}$$

Similarly, inequalities (3.8), together with the theorem on the minimum of the additional energy, also result in inequalities of the form (3.10).

The inequalities (3.7) or (3.10) can be interpreted as constraints imposed on the elastic properties of the medium under investigation provided that they are given for the auxiliary medium. Since usually it is λ that will be given from the very beginning, it is more convenient to rewrite inequalities (3.7) and (3.8) in the form of constraints imposed on the elastic properties of the auxiliary medium.

By (3.7) and (3.8), the parameters λ_c and μ_c should satisfy the inequalities

$$\frac{\lambda}{1 + k_1} \leqslant \lambda_c \leqslant \frac{\lambda}{1 - k_1}, \quad \frac{\mu}{1 + k_2} \leqslant \mu_c \leqslant \frac{\mu}{1 - k_2} \tag{3.11}$$

Taking account of the assumed boundedness of λ and μ we can conclude that the right sides of inequalities (3.11) result in the inequalities $(\varepsilon_c, \lambda_c, \varepsilon_c) \leqslant C_1$ and $(\sigma_c, \mu_c\sigma_c) \leqslant C_2$ which impose no substantial constraints on the selection of the parameters λ_c and μ_c. On the other hand, the left sides of the inequalities (3.11) determine the domains of values of the parameters λ_c and μ_c that satisfy the sufficient conditions for convergence.

Analyzing these inequalities, we arrive at the conclusion that under certain conditions, a domain of values of the parameter $\lambda_c = \mu_c^{-1}$ can exist that satisfy both left inequalities (3.11). In this case both expansions hold even for the fields ε and σ ((2.13) and (2.14)). In general, it can just be asserted that one of the parameters λ_c or μ_c, related by the equation $\lambda_c\mu_c = I$, can always be selected in such a way that the sufficient conditions for one of the series (2.14) to converge would be satisfied. Finding one of the fields ε or σ in the form of a convergent Neumann series enables the other to be calculated using the relations $\sigma = \lambda\varepsilon$ or $\varepsilon = \mu\sigma$.

If the parameters λ_c and μ_c are not related by the equation $\lambda_c\mu_c = I$, the constraints (3.11) are mutually independent. In this case, by selecting λ_c and μ_c from (3.11), we will have a basis for using both series (2.14).

4. Let us consider the elastic strain energy /2/

$$W = \int w dV \equiv V\{w\},\ 2w = \varepsilon_{ij}\sigma_{ij} = \bar{\varepsilon}_{ij}\bar{\sigma}_{ij}$$

where the braces denote averaging over the volume V. Keeping in mind the definition of the scalar product (2.1), we can hence write

$$2\{w\} = (\varepsilon, \sigma) = (\bar{\varepsilon}, \bar{\sigma}) \tag{4.1}$$

We can express the energy W in terms of the field ε_c To this end, we will rewrite (4.1) in the form

$$(\varepsilon, \sigma) = (\varepsilon_c, \sigma) + (\varepsilon', \sigma_c) \tag{4.2}$$

by using the equation $(\varepsilon', \sigma) = (\varepsilon', \sigma_c)$

Taking into account the equations $\sigma = \lambda\varepsilon = \lambda'\varepsilon + \lambda_c\varepsilon = \lambda'\varepsilon + \lambda_c\varepsilon' + \sigma_c$, we obtain from (4.1) and (4.2)

$$2\{w\} = (\varepsilon_c, \sigma_c) + (\varepsilon_c, \lambda'\varepsilon) + 2(\varepsilon', \sigma_c) \tag{4.3}$$

The first term on the right side of (4.3) is the elastic strain energy $2W_c/V$ in the comparison medium, while the latter is determined by external effects (surface and volume). To eliminate it, we introduce the potential energy /2/

$$U = W - \int \mathbf{ut} dS_2 - \int \mathbf{uf} dV \tag{4.4}$$

Subtracting the potential energy U_c from (4.4)

$$U' \equiv U - U_c = W' - \int \mathbf{u}'\mathbf{t} dS_2 - \int \mathbf{u}'\mathbf{f} dV,\ x' \equiv x - x_c$$

and passing here from a surface to a volume integral, we obtain

$$U' = W' - V(\varepsilon', \sigma) = W' - V(\varepsilon', \sigma_c)$$

Taking account of (4.3), we hence find

$$U'/V \equiv u' = {}^1/_2(\varepsilon_c, \lambda'\varepsilon) = {}^1/_2(\bar{\varepsilon}_c, \bar{\lambda}'\bar{\varepsilon}) \tag{4.5}$$

Furthermore, we use the solution of (2.13) for the field $\bar{\varepsilon}$ in the form $\bar{\varepsilon} = (I - \bar{X})^{-1}\bar{\varepsilon}_c$ which is valid if and only if the corresponding Neumann series diverges. Substituting it into (4.5), we obtain

$$2u' = (2\varepsilon_c', [\bar{Q} + \bar{q}]^{-1}\bar{\varepsilon}_0),\ \bar{q}\bar{\lambda}' = I \tag{4.6}$$

In addition to (4.6), a representation of the energy u' in terms of the field $\bar{\sigma}_c$ is possible. Instead of (4.3), we find similarly

$$2\{w\} = (\varepsilon_c, \sigma_c) + (\sigma_c, \mu'\sigma) + 2(\varepsilon_c, \sigma') \tag{4.7}$$

Transforming the potential energy (4.4) to the form

$$U = \int \mathbf{ut} dS_1 - W$$

and using (4.7), we arrive at the equation

$$u' = -{}^1/_2(\sigma_c, \mu'\sigma) = -{}^1/_2(\bar{\sigma}_c, \bar{\mu}'\bar{\sigma}) \tag{4.8}$$

Substituting into (4.8) the solution (2.13) for the field $\bar{\sigma}$ in the form $\bar{\sigma} = (I - \bar{Y})^{-1}\bar{\sigma}_c$ which holds in general, we shall have in place of (4.6)

$$2u' = -(\bar{\sigma}_c, [\bar{p} + \bar{P}]^{-1}\bar{\sigma}_c),\quad \bar{p}\bar{\mu}' = I \tag{4.9}$$

Formulas (4.6) and (4.9) enable us to calculate the energy u' by two equivalent means. Comparing (4.6) and (4.9), we note that

$$-(\bar{p} + \bar{P}) = \bar{Q} + \bar{q} \tag{4.10}$$

On the other hand, (4.10) follows from (2.10) and the relationship $\bar{p} + \bar{q} = -I$ which can easily be established.

5. Let the conditions for the series $\bar{a}$ and $\bar{b}$ to converge be satisfied. Then taking account of (2.13) and (2.14), respectively, we obtain from (4.5) and (4.8)

$$2u' = \sum_0^\infty l_k,\quad l_k \equiv (\varepsilon_c, \lambda'\varepsilon_k) = (\bar{\varepsilon}_c, \bar{\lambda}'\bar{\varepsilon}_k),\quad \bar{\varepsilon}_k \equiv \bar{X}^k\bar{\varepsilon}_c \tag{5.1}$$

$$2u' = -\sum_0^\infty m_k,\quad m_k \equiv (\sigma_c, \mu'\sigma_k) = (\bar{\sigma}_c, \bar{\mu}'\bar{\sigma}_k),\quad \bar{\sigma}_k \equiv \bar{Y}^k\sigma_c \tag{5.2}$$

$$(\varepsilon_k = \lambda_c^{-1/_2}\bar{\varepsilon}_k,\quad \sigma_k = \mu_c^{-1/_2}\bar{\sigma}_k)$$

The representation of the potential energy in the form of converging number series (5.1) and (5.2) enables us to obtain approximate solutions for u' as well as the limits $u_\pm'$ within which the value of u' lies.

We will first investigate the fixed-sign property of the series (5.1) and (5.2). The symmetry of the operators utilized as well as the properties (2.11) of the projection operators $\bar{P}$ and $\bar{Q}$ enable us to write

$$\bar{Q}\bar{X}=\bar{X},\ \bar{X}^{+}=-\bar{\lambda}'\bar{Q},\ \bar{P}\bar{Y}=\bar{Y},\ \bar{Y}^{+}=-\bar{\mu}'\bar{P} \tag{5.3}$$

By using (5.3) and the definitions (2.2) and (5.1), it can be shown that

$$\begin{gathered}(\bar{\varepsilon}_p,\bar{\lambda},\bar{\varepsilon}_q)=(\bar{\varepsilon}_{p-1},\bar{\lambda}'\bar{\varepsilon}_{q+1})=\ldots=(\bar{\varepsilon}_c,\bar{\lambda}'\bar{\varepsilon}_{p+q})\\ -(\bar{\varepsilon}_{p+1},\bar{\varepsilon}_{q+1})=(\bar{\varepsilon}_p,\bar{\lambda}'\bar{\varepsilon}_{q+1})=(\bar{\varepsilon}_c,\bar{\lambda}'\bar{\varepsilon}_{p+q+1});\ p,q\geqslant 0\end{gathered} \tag{5.4}$$

Correspondingly, (5.3) and the definition (2.2) and (5.2) yield

$$\begin{gathered}(\bar{\sigma}_p,\bar{\mu}'\bar{\sigma}_q)=(\bar{\sigma}_{p-1},\bar{\mu}'\bar{\sigma}_{q+1})=\ldots=(\bar{\sigma}_c,\bar{\mu}'\bar{\sigma}_{p+q})\\ -(\bar{\sigma}_{p+1},\bar{\sigma}_{q+1})=(\bar{\sigma}_p,\bar{\mu}'\bar{\sigma}_{q+1})=(\bar{\sigma}_c,\bar{\mu}'\bar{\sigma}_{p+q+1});\ p,q\geqslant 0\end{gathered} \tag{5.5}$$

The functionals l_k and m_k, defined by (5.1) and (5.2), can also be represented in the following form using (5.4) and (5.5):

$$l_{2k}=(\bar{\varepsilon}_k,\bar{\lambda}'\bar{\varepsilon}_k),\ l_{2k-1}=-(\bar{\varepsilon}_k,\bar{\varepsilon}_k),\ m_{2k}=(\bar{\sigma}_k,\bar{\mu}'\bar{\sigma}_k);\ m_{2k-1}=-(\bar{\sigma}_k,\bar{\sigma}_k) \tag{5.6}$$

It follows from (3.6) that the functionals l_{2k-1} and m_{2k-1} are negative, while the signs of the functionals l_{2k} and m_{2k} are determined by the value of the parameters λ_c and μ_c. Let λ_c and μ_c be such that $\lambda'\geqslant 0$ and $\mu'\leqslant 0$. In general, λ_c and μ_c cannot here be related by the equation $\lambda_c\mu_c=I$. We then have from (5.6) $l_{2k}\geqslant 0$ and $m_{2k}\leqslant 0$, i.e., series (5.1) and (5.2) will possess opposite sign-definiteness.

To obtain the boundaries for u', auxiliary inequalities are needed whose derivation is based on (5.6) taking the signs into account. Because of the positivity of scalar products of the form (f,f), the inequalities

$$\begin{gathered}l_{2k-1}+2l_{2k}+l_{2k+1}<0,\quad f\equiv\bar{\varepsilon}_k+\bar{\varepsilon}_{k+1},\bar{\varepsilon}_k\neq 0\\ m_{2k-1}+2m_{2k}+m_{2k+1}<0,\quad f\equiv\bar{\sigma}_k+\bar{\sigma}_{k+1},\sigma_k\neq 0\end{gathered} \tag{5.7}$$

hold.

Similarly, using the inequalities $(f,\bar{\lambda}'f)\geqslant 0$ and $(f,\bar{\mu}'f)\geqslant 0$, we can write

$$\begin{gathered}l_{2k}+2l_{2k+1}+l_{2k+2}\geqslant 0,\ \lambda'\geqslant 0\\ m_{2k}+2m_{2k+1}+m_{2k+2}\geqslant 0,\ \mu'\geqslant 0\end{gathered} \tag{5.8}$$

Summing inequalities (5.7) and (5.8) with respect to k between n and ∞, we obtain

$$2\sum_{2n-1}^{\infty}l_k<l_{2n-1}<0,\quad 2\sum_{2n-1}^{\infty}m_k<m_{2n-1}<0,\ n\geqslant 1 \tag{5.9}$$

$$2\sum_{2n}^{\infty}l_k\geqslant l_{2n}\geqslant 0,\ \lambda'\geqslant 0;\ \ 2\sum_{2n}^{\infty}m_k\geqslant m_{2n}\geqslant 0,\ \mu'\geqslant 0;\ n\geqslant 0 \tag{5.10}$$

We will first examine the case when $\lambda'>0$ and $\mu'<0$ when series (5.1) is sign-varying while (5.2) is sign-constant. Taking into account that

$$\sum_{k=0}^{\infty}x_k=\sum_{k=0}^{n-1}x_k+\sum_{k=n}^{\infty}x_k$$

and also inequalities (5.9) and (5.10), we find from (5.1) and (5.2), respectively

$$\frac{1}{2}\,l_{2n}<2u'-\sum_{0}^{2n-1}l_k<-\frac{1}{2}\,l_{2n-1},\ -2u'<\sum_{0}^{n}m_k \tag{5.11}$$

Therefore, utilization of the first expansion from (2.14) results in bilateral limits, and of the second in just the lower limit for the energy u'. We will have in the zeroth, first and second approximations from (5.11)

$$\begin{gathered}{}^1\!/_2 l_0<2u'<l_0,\ -2u'<m_0\\ l_0+l_1<2u'<l_0+{}^1\!/_2\,l_1,\ -2u'<m_0+m_1\\ l_0+l_1+{}^1\!/_2 l_2<2u'<l_0+l_1+l_2,\ -2u'<m_0+m_1+m_2\end{gathered} \tag{5.12}$$

Now, let $\lambda'<0$ and $\mu'>0$. In this case, series (5.1) will be sign-constant, and (5.2) sign-varying. Utilizing inequalities (5.9) and (5.10), we obtain from (5.1) and (5.2)

$$2u'<\sum_{0}^{n}l_k,\ \ \frac{1}{2}\,m_{2n}<-2u'-\sum_{0}^{2n-1}m_k<-\frac{1}{2}\,m_{2n-1} \tag{5.13}$$

It hence follows that the first expansion from (2.14) results only in the upper limit, and the second results in the bilateral limits for the energy u'. In the zeroth, first and second approximations we find from (5.13)

$$2u' < l_0,\ {}^1/_2 m_0 < -2u' < m_0$$
$$2u' < l_0 + l_1,\ m_0 + m_1 < -2u' < m_0 + {}^1/_2 m_1 \tag{5.14}$$
$$2u' < l_0 + l_1 + l_2,\ m_0 + m_1 + {}^1/_2 m_2 < -2u' < m_0 + m_1 + m_2$$

It is seen from (5.11)−(5.14) that we have bilaterial limits for the energy u' in any case. To obtain them it is necessary here to use the first expansion from (2.14) if $\lambda' > 0$ and the second if $\mu' > 0$.

If the selection of the parameters λ_c and μ_c is not constrained by the requirement of sign-definiteness of λ' and μ', we will have the following boundaries instead of (5.11) and (5.13)

$$-\sum_0^{2n-1} m_k + \frac{1}{2} m_{2n-1} < 2u' < \sum_0^{2n-1} l_k - \frac{1}{2} l_{2n-1} \tag{5.15}$$

constructed on the basis of inequalities (5.7) which hold in general.

6. Inequalities analogous to (3.10) can be set up in the problem of estimating the error due to replacing the solution of problem (1.1) by the solution $\mathbf{u}_c$ of the problem obtained from (1.1) by replacing L by L_c /8/. Assuming the operators L and L_c to be positive definite and "semi-convergent" the following inequalities /7, 8/ turn out to be valid

$$\alpha |\mathbf{u}|_c^2 \leqslant |\mathbf{u}|^2 \leqslant \beta |\mathbf{u}|_c^2,\ |\mathbf{u}|^2 = (\mathbf{u}, L\mathbf{u}),\ |\mathbf{u}|_c^2 = (\mathbf{u}_c, L_c \mathbf{u}_c) \tag{6.1}$$

where the fields $\mathbf{u}, \mathbf{u}_c$ satisfy homogeneous boundary conditions. The quantities α and β are found from the solution of the generalized eigennumber problem for the operator L of the form $L\mathbf{u} - \varkappa L_c \mathbf{u} = 0$, and respectively equal to $\alpha = \inf \varkappa'$, $\beta = \sup \varkappa''$, where $\varkappa'$ is the minimum and $\varkappa''$ the maximum of the eigennumbers /7/. No constraints have here been imposed on α and β and, therefore, on L and L_c.

Unlike (6.1), inequalities (3.10) take account of the fact that the first series in (2.14), in which the operator Q is constructed using L_c, converges. Consequently, α and β cannot be arbitrary but must satisfy certain constraints. In particular, the following inequalities result from (3.10):

$$0 < 1 - k \leqslant \alpha \leqslant \beta \leqslant 1 + k < 2, \qquad k \equiv k_1 \tag{6.2}$$

The possibility should also be pointed out of another approach to the solution of problem (1.1), namely, the application of an iteration method directly to analyze the displacement field $\mathbf{u}$. However, taking account of the boundary conditions in the form (1.2) complicates the construction considerably. In the special case of the first boundary value problem (the Dirichlet problem), the solution for the field $\mathbf{u}$ can be represented in the form

$$\mathbf{u} = \mathbf{u}_c + ML'\mathbf{u},\ L' = L - L_c \tag{6.3}$$

where M is the operator whose kernel is Green's tensor G. Equation (6.3) is similar in form to (1.13) and (2.9) as they are all inhomogeneous integral equations of the second kind. Solving it by iterations, we can write in the same was as for (1.15) and (2.14)

$$\mathbf{u} = \sum_{k=0}^{\infty} (ML')^k \mathbf{u}_c \tag{6.4}$$

Series (6.4) converges provided that $\| ML' \| \leqslant k < 1$, which can be written in the form

$$\| ML' \| \leqslant \| M \| \| L' \| < k < 1 \tag{6.5}$$

Inequalities similar to (6.5) were utilized in /8/ to find the stability criterion for the approximate solution of equations of type (1.1). The convergence conditions (6.5) for the series (6.4) cannot, however be converted to the form (3.7), and even more (3.8). To obtain inequalities of the type (6.5) but containing the pliabilities μ and μ_c, it is necessary to go from the equilibrium equation (1.1) to the incompatibility equation /1, 2/. The reasoning presented agrees with the remark made by S.G. Mikhlin /7/ about the passage from constant coefficients λ in (1.1) to variable coefficients. It is indicated that such an approach produces serious difficulties in the utilization of such methods as the method of integral equations or the method of Green's function, especially when seeking elementary particular solutions.

A method is proposed in a published paper for solving a fairly broad class of linear boundary value problems in the form (1.1) and (1.2). It is based on the above-mentioned methods of Green's function and integral equations. Some of the difficulties originating here are overcome by introducing the projection operators $\bar{P}$ and $\bar{Q}$. Utilization of an auxiliary medium enables a solution to be found to such problems, which cannot be solved (or their solution is quite difficult) by direct methods. This holds particularly in the case of inhomogeneous and anisotropic media.

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ON A PERIODIC MIXED PROBLEM FOR A STRIP*

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The periodic problem of the action of rigid stamps on a strip is considered. The foundations of the stamps are assumed to be arbitrarily convex and symmetric about their vertical axes. The problem is reduced to dual summation equations by a traditional method. Two cases are studied: the corners of the stamp press on the strip (the width of the contact area is known), and the corners of the stamp do not reach the strip (the width of the contact area is unknown). The solution for stamps with flat bases follows as a special case from the solution obtained. This is simultaneously the solution (apart from sign and notation) of a certain doubly-periodic problem for a plane with slits.

The problem under consideration has been studied by other methods in /1–3/.

1. The domain of the strip to be studied lies in the complex $z = x + iy$ plane (see Fig. 1 on which the base of the stamps is shown flat for simplicity). The stamps acting on a strip from both sides have identical width and are arranged symmetrically with period $2b$. Therefore, the problem is periodic, and, consequently, we refer all reasoning to the fundamental period $-b \leqslant x \leqslant b$.

We will write the boundary conditions forthe upper boundary $y = a$

$$v = -f(x), \; |x| < c \qquad (1.1)$$
$$Y_y = 0, \; c < |x| \leqslant b$$
$$X_y = 0, \; |x| \leqslant b$$

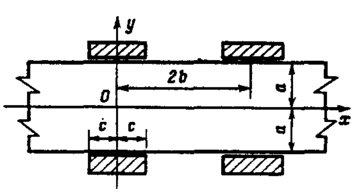

Fig.1

The form of the function $f(x)$ will be indicated below. We denote the pressure under the stamps by $Y_{y*}(x)$ then the load $Y_{ya}(x)$ on the faces of the strip can be represented as follows:

$$Y_{ya}(x) = \begin{cases} Y_{y*}(x), & |x| < c \\ 0, & c < |x| \leqslant b \end{cases}; \; X_{ya}(x) = 0, \quad |x| \leqslant b \qquad (1.2)$$

We expand the periodic load $Y_{ya}(x)$ in a Fourier series

$$Y_{ya}(x) = \sum_{n=0}^{\infty} a_n \cos \varkappa_n x, \quad \varkappa_n = \frac{n\pi}{b} \qquad (1.3)$$

$$a_0 = \frac{1}{b} \int_0^b Y_{ya}(x)\, dx, \quad a_n = \frac{2}{b} \int_0^b Y_{ya}(x) \cos \varkappa_n r\, dx$$

We now rewrite the boundary conditions (1.1) by using the Kolosov-Muskhelishvili potentials

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